Hidden symmetries and Killing tensors on curved spaces

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Higher order symmetries corresponding to Killing tensors are investigated. The intimate relation between Killing-Yano tensors and non-standard supersymmetries is pointed out. In the Dirac theory on curved spaces, Killing-Yano tensors generate Dirac type operators involved in interesting algebraic structures as dynamical algebras or even infinite dimensional algebras or superalgebras. The general results are applied to space-times which appear in modern studies. One presents the infinite dimensional superalgebra of Dirac type operators on the 4-dimensional Euclidean Taub-NUT space that can be seen as a twisted loop algebra. The existence of the conformal Killing-Yano tensors is investigated for some spaces with mixed Sasakian structures.

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1. INTRODUCTION

One of the key concepts in physics is that of symmetries, Noether's theorem giving a correspondence between symmetries and conserved quantities. For the geodesic motions on a space-time the usual conserved quantities are related to the isometries which correspond to Killing vectors. Sometimes a space-time could admit higher order symmetries described by symmetric Killing tensors, called Stäckel-Killing (S-K). These symmetries are known as *hidden symmetries* and the typical example is the Runge-Lenz vector in the Kepler/Coulomb problem. The corresponding conserved quantities are quadratic, or, more general, polynomial in momenta. Their existence guarantees the integrability of the geodesic motions and is

intimately related to separability of Hamilton-Jacobi (see, e. g. [1]) and the Klein-Gordon equation at the quantum level [2].

The next most simple objects that can be studied in connection with the symmetries of a manifold after the S-K tensors are the Killing-Yano tensors (K-Y) [3]. It was observed [4] that a K-Y tensor generates additional *supercharges* in the dynamics of pseudo-classical spinning particles being the natural geometrical objects to be coupled with the fermionic degrees of freedom [4, 5]. In this way it was realized the significant connection between K-Y tensors and *non-standard supersymmetries*. Passing to quantum Dirac equation it was discovered [6] that K-Y tensors generate conserved *non-standard* Dirac operators which commute with the *standard* one.

The conformal extension of the Killing tensor equation determines the conformal Killing tensors [7] which define first integrals of the null geodesic equation. Investigation of the properties of higher-dimensional space-times has pointed out the role of the conformal K-Y (CKY) tensors to generate background metrics with black-hole solutions (see, e. g. [8]).

The aim of this paper is to investigate a few examples of curved spaces endowed with special structures admitting K-Y tensors which could be relevant in the theories of modern physics.

The first example is represented by the 4-dimensional Euclidean Taub-Newman-Unti-Tamburino (Taub-NUT) space. The motivation to carry out this example is twofold. First of all, in the Taub-NUT geometry there are known to exist four K-Y tensors [9]. From this point of view the Taub-NUT manifold is an exceedingly interesting space to exemplify the effective construction of the conserved quantities in terms of geometric ones. On the other hand, the Taub-NUT geometry is involved in many modern studies in physics.

In the second example we investigate the existence of CKY tensors in higher dimensional space-times [10]. Investigations of the properties of space-times of higher dimensions ($D > 4$) have recently attracted considerable attention as a result of their appearance in theories of unification such as string and M theories. Versions of $M-$theory could be formulated in space-times with various number of time dimensions giving rise to exotic space-time signatures. The $M-$theory in $10+1$ dimensions is linked via dualities to a M^* theory in $9+2$ dimensions and a M' theory in $6+5$ dimensions. Various limits of these will give rise to $IIA-$ and $IIB-$like string theories in many variants of dimensions and signatures [11].

The plan of the paper is as follows: In section 2 we present the Killing tensors which

generalize the Killing vectors. In section 3 we describe the Dirac-type operators generated by K-Y tensors and the general results are applied to the 4-dimensional Euclidean Taub-NUT space [12, 13]. The CKY tensors on manifolds with mixed 3−structures are presented in the last section 4. Some details concerning the geometrical properties of these metrics are summarized in the Appendices.

2. KILLING VECTOR FIELDS AND THEIR GENERALIZATIONS

Let (M, g) be a semi-Riemannian manifolds. A vector field X on M is said to be a Killing vector field if the Lie derivative with respect to X of the metric g vanishes.

Killing vector fields can be generalized to conformal Killing vector fields [3], i.e. vector fields with a flow preserving a given conformal class of metrics. A natural generalization of conformal Killing vector fields is given by the CKY tensors [14].

Definition 1 *A CKY tensor of rank p on a semi-Riemannian manifold (M, g) is a p-form f which satisfies:*

$$\nabla_X f = \frac{1}{p+1} X \lrcorner f - \frac{1}{n-p+1} X^* \wedge d^* f \,, \tag{1}$$

for any vector field X on M.

Here ∇ is the Levi-Civita connection of g, n is the dimension of M, X^* is the 1-form dual to the vector field X with respect to the metric g, $\lrcorner$ is the operator dual to the wedge product and d^* is the adjoint of the exterior derivative d. If f is co-closed in (1), then we obtain the definition of a K-Y tensor (introduced by Yano [3]). We can easily see that for $p = 1$, they are dual to Killing vector fields.

A K-Y tensor can be characterized in several ways. In a equivalent manner a differential $p-$form f is called a K-Y tensor if its covariant derivative $\nabla_\lambda f_{\mu_1 \ldots \mu_p}$ is totally antisymmetric. As a consequence of the antisymmetry a K-Y tensor satisfy the equation $\nabla_{(\lambda} f_{\mu_1) \ldots \mu_p} = 0$. Let us remark that for covariantly constant K-Y tensors each term of the l. h. s. of this equation vanishes. The covariantly constant K-Y tensors represent a particular class of K-Y tensors and they play a special role in the theory of Dirac operators as it will be seen in section 3.

We mention that K-Y tensors are also called Yano tensors or Killing forms, and CKY tensors are sometimes referred as conformal Yano tensors, conformal Killing forms or twistor forms [15–17].

For generalizations of the Killing vectors one might also consider higher order symmetric tensors.

Definition 2 *A symmetric tensor of $K_{\mu_1\ldots\mu_r}$ of rank $r > 1$ satisfying a generalized Killing equation $\nabla_{(\lambda}K_{\mu_1\ldots\mu_r)} = 0$ is called a S-K tensor.*

The relevance in physics of the S-K tensors is given by the following proposition which could be easily proved:

Proposition 1 *A symmetric tensor K on M is a S-K tensor iff the quantity $K = K_{\mu_1\ldots\mu_r}\dot{s}^{\mu_1}\cdots\dot{s}^{\mu_r}$ is constant along every geodesic s in M.*

Here the over-dot denotes the ordinary proper time derivative and the proposition ensures that K is a first integral of the geodesic equation.

These two generalizations S-K and K-Y of the Killing vectors could be related. Let $f_{\mu_1\ldots\mu_p}$ be a K-Y tensor, then the tensor field $K_{\mu\nu} = f_{\mu\mu_2\ldots\mu_p}f^{\mu_2\ldots\mu_p}{}_{\nu}$ is a S-K tensor and it sometimes refers to this S-K tensor as the associated tensor with f. However, the converse statement is not true in general: not all S-K tensors of rank 2 are associated with a K-Y tensor.

3. DIRAC-TYPE OPERATORS

For a quantum relativistic description of a spin-1/2 particle on a curved space we use the *standard* Dirac operator

$$D_s = i\gamma^\mu\nabla_\mu\,, \tag{2}$$

where ∇_μ are the spin covariant derivatives including spin-connection, while γ^μ are the standard Dirac matrices carrying natural indices.

We note that for any isometry with a Killing vector R^μ there is an appropriate operator

$$X_k = -i(R^\mu\nabla_\mu - \frac{1}{4}\gamma^\mu\gamma^\nu R_{\mu;\nu})\,, \tag{3}$$

which commutes with D_s. Moreover each K-Y tensor $f_{\mu\nu}$ produces a *non-standard* Dirac operator of the form [6]

$$D_f = i\gamma^\mu(f_\mu{}^\nu\nabla_\nu - \frac{1}{6}\gamma^\nu\gamma^\rho f_{\mu\nu;\rho})\,, \tag{4}$$

which anticommutes with the standard Dirac operator D_s and can be involved in new types of genuine or hidden (super)symmetries.

3.1. Covariantly constant K-Y tensors

Remarkable superalgebras of Dirac-type operators can be produced by special second-order K-Y tensors that represent square roots of the metric tensor [18–20].

Definition 3 *The non-singular real or complex-valued K-Y tensor f of rank 2 defined on M which satisfies*

$$f^{\mu}{}_{\alpha} f_{\mu\beta} = g_{\alpha\beta}\,, \tag{5}$$

is called an unit root of the metric tensor of M, or simply an unit root of M.

Let us observe that any unit root K-Y tensor is covariantly constant [18], i. e. $f_{\mu\nu;\sigma} = 0$.

It is worthy to be noted that the covariantly constant K-Y tensors give rise to Dirac-type operators of the form (4) connected with the standard Dirac operators as follows:

Theorem 1 *The Dirac-type operator D_f produced by the K-Y tensor f satisfies the condition $(D_f)^2 = D_s^2$ iff f is an unit root.*

Proof: The arguments of Ref. [18] show that the condition from the theorem is equivalent with (5) f being a covariantly constant K-Y tensor. ∎

3.2. Dirac operators on Euclidean Taub-NUT space

To make things more specific let us consider the Euclidean Taub-NUT space (see Appendix A) which is hyper-Kähler and possesses many non-standard symmetries expressed in terms of four K-Y tensors and three S-K tensors.

From the covariantly constant K-Y tensors f^i (A.2), using prescription (4), we can construct three Dirac-type operators $D^{(i)}$ which anticommute with standard Dirac operator D_s (2). It is convenient to define [21] $Q_i = iH^{-1}D^{(i)}$ where $H = -\gamma^0 D_s$ is the massless Hamiltonian operator. These operators form a representation of the quaternionic units: $Q_i\, Q_j = \delta_{ij} I + i\varepsilon_{ijk} Q_k$.

On the other hand Dirac-type operator constructed from the K-Y tensor f^Y (A.3) is D^Y and again it is convenient to define a new operator $Q^Y = HD^Y$.

The conserved Runge-Lenz operator of the Dirac theory is

$$\mathcal{K}_i = \frac{\mu}{4}\{Q^Y,\, Q_i\} + \frac{1}{2}(\mathcal{B} - P_4)Q_i - \mathcal{J}_i P_4\,, \tag{6}$$

where $\mathcal{B}^2 = {P_4}^2 - H^2,\ J_i\,,(i=1,2,3)$ are the components of the total angular momentum, while $P_4 = -i\partial_4$ corresponding to the fourth Cartesian coordinate $x^4 = -4m(\chi + \varphi)$.

The operators J_i and $\mathcal{K}_i$ are involved in the following system of commutation relations:

$$[\mathcal{J}_i,\ \mathcal{J}_j] = i\varepsilon_{ijk}\mathcal{J}_k,\ [\mathcal{J}_i,\ \mathcal{K}_j] = i\varepsilon_{ijk}\mathcal{K}_k\,, \qquad [\mathcal{K}_i,\ \mathcal{K}_j] = i\varepsilon_{ijk}\mathcal{J}_k\mathcal{B}^2\,, \tag{7}$$

and commute with the operators Q_i

$$[\mathcal{J}_i,\ Q_j] = i\varepsilon_{ijk}Q_k\,, \qquad [\mathcal{K}_i,\ Q_j] = i\varepsilon_{ijk}Q_k\mathcal{B}\,. \tag{8}$$

The algebra (7) does not close as a finite Lie algebra because of the factor $\mathcal{B}^2$. In the standard treatment one concentrates on individual subspaces of the whole Hilbert space which belong to definite eigenvalues of $\mathcal{B}^2$. This is similar to the dynamical algebra of the hydrogen atom which can be identified in a natural way with an infinite dimensional twisted loop algebra [22].

The dynamical algebras of the Dirac theory have to be obtained by replacing this operator $\mathcal{B}^2$ with its eigenvalue $q^2 - E^2$ and rescaling the operators $\mathcal{K}_i$. The same kind of problems appears for the anticommutators involving the fermionic operators Q_i and Q^Y. In what follows, in order to keep the presentation as simple as possible, we shall only give the briefest account of the algebra of operators connected with hidden symmetries in the bosonic sector. For the algebra of operators from the fermionic sector the reader should consult [23].

In the bosonic sector of conserved operators let us define the new operators "absorbing" the operator $\mathcal{B}$ by assigning grades to each operator [24]:

$$A^i_{2n} := \mathcal{J}_i\mathcal{B}^n\,, \quad B^i_{2n+2} := \mathcal{K}_i\mathcal{B}^n\,, \tag{9}$$

for any $n = 0, 1, 2....$ The algebra of these operators can be seen as an infinite dimensional twisted loop algebra of the Kac-Moody type. In this way the commutation relations of the bosonic sector are given by a Kac-Moody type algebra:

$$\begin{aligned}
\left[A^i_{2n}, A^j_{2m}\right] &= i\varepsilon_{ijk}A^k_{2(n+m)}\,,\\
\left[A^i_{2n}, B^j_{2m+2}\right] &= i\varepsilon_{ijk}B^k_{2(n+m+1)}\,,\\
\left[B^i_{2n+2}, B^j_{2m+2}\right] &= i\varepsilon_{ijk}A^k_{2(n+m+2)}\,.
\end{aligned} \tag{10}$$

4. CKY TENSORS ON MANIFOLDS WITH MIXED 3-STRUCTURES

An almost para-hypercomplex structure on a smooth manifold M is a triple $H = (J_\alpha)_{\alpha=\overline{1,3}}$, where J_1 is an almost complex structure on M and J_2, J_3 are almost product structures on M, satisfying: $J_1J_2J_3 = -Id$. In this case (M, H) is said to be an almost para-hypercomplex manifold.

A semi-Riemannian metric g on (M, H) is said to be para-hyperhermitian if it satisfies $g(J_\alpha X, J_\alpha Y) = \epsilon_\alpha g(X, Y),\ \alpha \in \{1, 2, 3\}$ for all $X, Y \in \Gamma(TM)$, where $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$. In this case, (M, g, H) is called an almost para-hyperhermitian manifold. Moreover, if each J_α is parallel with respect to the Levi-Civita connection of g, then (M, g, H) is said to be a para-hyper-Kähler manifold.

Theorem 2 *Let (M, g) be a semi-Riemannian manifold. Then the following five assertions are mutually equivalent:*

(1) (M, g) admits a mixed 3-Sasakian structure.

(2) The cone $(C(M), \overline{g}) = (M \times \mathbf{R}_+, dr^2 + r^2 g)$ admits a para-hyper-Kähler structure.

(3) There exists three orthogonal Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ on M, with ξ_1 unit space-like vector field and ξ_2, ξ_3 unit time-like vector fields satisfying

$$[\xi_\alpha, \xi_\beta] = -2\epsilon_\gamma \xi_\gamma, \tag{11}$$

where (α, β, γ) is an even permutation of (1,2,3) and $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$, such that the tensor fields ϕ_α of type (1,1), defined by: $\phi_\alpha X = -\epsilon_\alpha \nabla_X \xi_\alpha,\ \alpha \in \{1, 2, 3\}$, satisfies the conditions (B.1), (B.2) and (B.3).

(4) There exists three orthogonal Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ on M, with ξ_1 unit space-like vector field and ξ_2, ξ_3 unit time-like vector fields satisfying (11), such that:

$$R(X, \xi_\alpha)Y = g(\xi_\alpha, Y)X - g(X, Y)\xi_\alpha,\ \alpha \in \{1, 2, 3\}, \tag{12}$$

where R is the Riemannian curvature tensor of the Levi-Civita connection ∇ of g.

(5) There exists three orthogonal Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ on M, with ξ_1 unit space-like vector field and ξ_2, ξ_3 unit time-like vector fields satisfying (11), such that the sectional curvature of every section containing ξ_1, ξ_2 or ξ_3 equals 1.

Proof: (1) $\Rightarrow$ (2) If M^{4n+3} is a manifold endowed with a mixed 3-Sasakian structure (see Appendix B) $((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$, then we can define a para-hyper-Kähler structure

$\{J_\alpha\}_{\alpha=\overline{1,3}}$ on the cone $(C(M),\overline{g}) = (M\times\mathbf{R}_+, dr^2+r^2g)$, by $J_\alpha X = \phi_\alpha X - \eta_\alpha(X)\Phi$, $J_\alpha\Phi = \xi_\alpha$ for any $X\in\Gamma(TM)$ and $\alpha\in\{1,2,3\}$, where $\Phi = r\partial_r$ is the Euler field on $C(M)$.

(2) $\Rightarrow$ (1) If the cone $(C(M),\overline{g}) = (M\times\mathbf{R}_+, dr^2+r^2g)$ admits a para-hyper-Kähler structure $\{J_\alpha\}_{\alpha=\overline{1,3}}$, then we can identify M with $M\times\{1\}$ and we have a mixed 3-Sasakian structure $((\phi_\alpha,\xi_\alpha,\eta_\alpha)_{\alpha=\overline{1,3}},g)$ on M given by $\xi_\alpha = J_\alpha(\partial_r)$, $\phi_\alpha X = -\epsilon_\alpha\nabla_X\xi_\alpha$, $\eta_\alpha(X) = g(\xi_\alpha,X)$, for any $X\in\Gamma(TM)$ and $\alpha\in\{1,2,3\}$.

(2) $\Leftrightarrow$ (3) This equivalence is clear (see also [25]).

(3) $\Leftrightarrow$ (4) This equivalence follows from direct computations.

(4) $\Leftrightarrow$ (5) This equivalence follows using the formula of the sectional curvature. ■

From the above Theorem we can easily obtain the next properties (see also [17]).

Corollary 1 *Let M^{4n+3} be a manifold endowed with a mixed 3-Sasakian structure $((\phi_\alpha,\xi_\alpha,\eta_\alpha)_{\alpha=\overline{1,3}},g)$. Then:*

(1) ξ_1 is unit space-like Killing vector field and ξ_2,ξ_3 are unit time-like Killing vector fields on M;

(2) η_1,η_2,η_3 are CKY tensors of rank 1 on M;

(3) $d\eta_1,d\eta_2,d\eta_3$ are CKY tensors of rank 2 on M;

(4) M admits K-Y tensors of rank $(2k+1)$, for $k\in\{0,1,...,2n+1\}$.

Corollary 2 *Let M^{4n+3} be a manifold endowed with a mixed 3-Sasakian structure $((\phi_\alpha,\xi_\alpha,\eta_\alpha)_{\alpha=\overline{1,3}},g)$. Then the distribution spanned by $\{\xi_1,\xi_2,\xi_3\}$ is integrable and defines a 3-dimensional Riemannian foliation on M, having totally geodesic leaves of constant curvature 1.*

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APPENDIX A. EUCLIDEAN TAUB-NUT SPACE

Let us consider the Taub-NUT space [12, 13] and the chart with Cartesian coordinates $x^\mu(\mu,\nu=1,2,3,4)$ having the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f(r)(d\vec{x})^2 + \frac{g(r)}{16m^2}(dx^4 + A_i dx^i)^2\,, \tag{A.1}$$

where $\vec{x}$ denotes the three-vector $\vec{x} = (r, \theta, \varphi)\,, (d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ and $\vec{A}$ is the gauge field of a monopole $\mathrm{div}\vec{A} = 0\,, \quad \vec{B} = \mathrm{rot}\,\vec{A} = 4m\frac{\vec{x}}{r^3}$. The real number m is a parameter of the theory which enter in the form of the functions $f(r) = g^{-1}(r) = \frac{4m+r}{r}$ and the so called NUT singularity is absent if x^4 is periodic with period $16\pi m$. Sometimes it is convenient to make the coordinate transformation $x^4 = -4m(\chi + \varphi)$, with $0 \leq \chi < 4\pi$.

In the Taub-NUT geometry there are four Killing vectors [9]. Three Killing vectors correspond to the invariance of the metric (A.1) under spatial rotations, obeying an $SU(2)$ algebra, while the fourth generates the $U(1)$ of χ translations, commuting with the other Killing vectors.

On the other hand in the Taub-NUT geometry there are known to exist four K-Y tensors of valence 2. The first three

$$f^i = 8m(d\chi + \cos\theta d\varphi) \wedge dx_i - \epsilon_{ijk}(1 + \frac{4m}{r})dx_j \wedge dx_k\,, i, j, k = 1, 2, 3\,, \tag{A.2}$$

are covariantly constant, i.e. $\nabla_\mu f^i_{\nu\lambda} = 0$. The f^i define three anticommuting complex structures of the Taub-NUT manifold, their components realizing the quaternion algebra $f^i f^j + f^j f^i = -2\delta_{ij},\ f^i f^j - f^j f^i = -2\varepsilon_{ijk} f^k$. The existence of these K-Y tensors is linked to the hyper-Kähler geometry of the manifold and shows directly the relation between the geometry and the $N = 4$ supersymmetric extension of the theory [4].

The fourth K-Y tensor is

$$f^Y = 8m(d\chi + \cos\theta d\varphi) \wedge dr + 4r(r + 2m)(1 + \frac{r}{4m}) \sin\theta d\theta \wedge d\varphi\,, \tag{A.3}$$

having a non-vanishing covariant derivative $f^Y_{r\theta;\varphi} = 2(1 + \frac{r}{4m})r\sin\theta$.

APPENDIX B. MANIFOLDS WITH MIXED 3-STRUCTURES

Let M be a differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on M such that: $\phi^2 = -\epsilon I + \eta \otimes \xi,\ \eta(\xi) = \epsilon$. If $\epsilon = 1$ then (ϕ, ξ, η) is said to be an almost contact structure on M (see [26]), and if $\epsilon = -1$ then (ϕ, ξ, η) is said to be an almost paracontact structure on M (see [27]).

Definition 4 *[28] Let M be a differentiable manifold which admits an almost contact structure (ϕ_1, ξ_1, η_1) and two almost paracontact structures (ϕ_2, ξ_2, η_2) and (ϕ_3, ξ_3, η_3), satisfying*

the following conditions:

$$\eta_\alpha(\xi_\beta) = 0\,, \forall \alpha \neq \beta\,, \qquad \phi_\alpha(\xi_\beta) = -\phi_\beta(\xi_\alpha) = \epsilon_\gamma \xi_\gamma,$$

$$\eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha = \epsilon_\gamma \eta_\gamma\,, \qquad \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta = \epsilon_\gamma \phi_\gamma,$$

where (α, β, γ) *is an even permutation of (1,2,3) and* $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$.

Then the manifold M *is said to have a mixed 3-structure* $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$.

Definition 5 *If a manifold* M *with a mixed 3-structure* $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ *admits a semi-Riemannian metric* g *such that:*

$$g(\phi_\alpha X, \phi_\alpha Y) = \epsilon_\alpha g(X,Y) - \eta_\alpha(X)\eta_\alpha(Y),\; g(X, \xi_\alpha) = \eta_\alpha(X)\,,$$

for all $X, Y \in \Gamma(TM)$ *and* $\alpha = 1, 2, 3$, *then we say that* M *has a metric mixed 3-structure and* g *is called a compatible metric. Moreover, if* $(\phi_1, \xi_1, \eta_1, g)$ *is a Sasakian structure, i.e. (see [26]):*

$$(\nabla_X \phi_1)Y = g(X,Y)\xi_1 - \eta_1(Y)X\,, \tag{B.1}$$

and $(\phi_2, \xi_2, \eta_2, g)$, $(\phi_3, \xi_3, \eta_3, g)$ *are LP-Sasakian structures, i.e. (see [27]):*

$$(\nabla_X \phi_2)Y = g(\phi_2 X, \phi_2 Y)\xi_2 + \eta_2(Y)\phi_2^2 X, \tag{B.2}$$

$$(\nabla_X \phi_3)Y = g(\phi_3 X, \phi_3 Y)\xi_3 + \eta_3(Y)\phi_3^2 X, \tag{B.3}$$

then $((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ *is said to be a mixed Sasakian 3-structure on* M.

It is easy to see that any manifold M with a mixed 3-structure admits a compatible semi-Riemannian metric g. Moreover, the signature of g is $(2n+1, 2n+2)$ and the dimension of the manifold M is $4n+3$. The main property of a manifold endowed with a mixed 3-Sasakian structure is the following (see [29]):

Theorem 3 *Any* $(4n+3)-$*dimensional manifold endowed with a mixed 3-Sasakian structure is an Einstein space with Einstein constant* $\lambda = 4n + 2$.

Concerning the symmetric Killing tensors let us note that D.E. Blair studied in [30] the almost contact manifold with Killing structure tensors. He assumed that M has an almost contact metric structure (ϕ, ξ, η, g) such that ϕ and η are Killing. Then he proved that if (ϕ, ξ, η, g) is normal, it is a cosymplectic structure.

For a mixed 3-structure with a compatible semi-Riemannian metric g, we have the following result:

Proposition 2 *Let (M, g) be a semi-Riemannian manifold. If (M, g) has a mixed 3-Sasakian structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$, then $(\phi_\alpha)_{\alpha=\overline{1,3}}$ cannot be Killing tensor fields.*

Proof: If $(\phi_1, \xi_1, \eta_1, g)$ is a Sasakian structure, from (B.1) we obtain: $(\nabla_X \phi_1)X = g(X, X)\xi_1 \neq 0$ for any non-lightlike vector field X orthogonal to ξ_1.

For a LP-Sasakian structure $(\phi_2, \xi_2, \eta_2, g)$, from (B.2) we have:

$$(\nabla_X \phi_2)X = g(\phi_2 X, \phi_2 X)\xi_2 \neq 0\,, \tag{B.4}$$

for any non-lightlike vector field X orthogonal to ξ_2. ■

Theorem 4 *Let (M, g) be a semi-Riemannian manifold. If (M, g) admits a mixed 3-Sasakian structure, then any conformal Killing vector field on (M, g) is a Killing vector field.*

Proof: A vector field X on M is conformal Killing iff $L_X g = f \cdot g$ for $f \in C^\infty(M, \mathbb{R})$ and we have $(L_X g)(\xi_\alpha, \xi_\alpha) = f g(\xi_\alpha, \xi_\alpha) = \epsilon_\alpha f$. But, by the Lie operator's properties,

$$\begin{aligned}(L_X g)(\xi_\alpha, \xi_\alpha) &= X g(\xi_\alpha, \xi_\alpha) - 2g(L_X \xi_\alpha, \xi_\alpha) = -2g([X, \xi_\alpha], \xi_\alpha)\\ &= -2g(\nabla_X \xi_\alpha, \xi_\alpha) + 2g(\nabla_{\xi_\alpha} X, \xi_\alpha) = 2\epsilon_\alpha g(\phi_\alpha X, \xi_\alpha) - 2g(X, \nabla_{\xi_\alpha}\xi_\alpha) = 0\,,\end{aligned} \tag{B.5}$$

because $\phi_\alpha X \perp \xi_\alpha$ ($\alpha \in \{1, 2, 3\}$) and $\nabla_{\xi_\alpha}\xi_\alpha = 0$.

Consequently, $f = \epsilon_\alpha (L_X g)(\xi_\alpha, \xi_\alpha) = 0$, so that $L_X g = 0$, i.e. X is Killing vector field. ■

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